$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots=\frac{\pi^2}{12}$

$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6}$

Mary L. Boas: "Mathematical methods in the Physical Sciences" Third edition.

CHAPTER 1

Infinite Series, Power Series

1. THE GEOMETRIC SERIES

As a simple example of many of the ideas involved in series, we are going to consider the geometric series. You may recall that in a geometric progression we multiply each term by some fixed number to get the next term. For example, the *sequences*

(1.1a) $$2,\ 4,\ 8,\ 16,\ 32, \ldots,$$

(1.1b) $$1,\ \tfrac{2}{3},\ \tfrac{4}{9},\ \tfrac{8}{27},\ \tfrac{16}{81}, \ldots,$$

(1.1c) $$a,\ ar,\ ar^2,\ ar^3, \ldots,$$

are geometric progressions. It is easy to think of examples of such progressions. Suppose the number of bacteria in a culture doubles every hour. Then the terms of (1.1a) represent the number by which the bacteria population has been multiplied after 1 hr, 2 hr, and so on. Or suppose a bouncing ball rises each time to $\frac{2}{3}$ of the height of the previous bounce. Then (1.1b) would represent the heights of the successive bounces in yards if the ball is originally dropped from a height of 1 yd.

In our first example it is clear that the bacteria population would increase without limit as time went on (mathematically, anyway; that is, assuming that nothing like lack of food prevented the assumed doubling each hour). In the second example, however, the height of bounce of the ball decreases with successive bounces, and we might ask for the total distance the ball goes. The ball falls a distance 1 yd, rises a distance $\frac{2}{3}$ yd and falls a distance $\frac{2}{3}$ yd, rises a distance $\frac{4}{9}$ yd and falls a distance $\frac{4}{9}$ yd, and so on. Thus it seems reasonable to write the following expression for the total distance the ball goes:

(1.2) $$1+2\cdot\tfrac{2}{3}+2\cdot\tfrac{4}{9}+2\cdot\tfrac{8}{27}+\cdots=1+2\left(\tfrac{2}{3}+\tfrac{4}{9}+\tfrac{8}{27}+\cdots\right),$$

where the three dots mean that the terms continue as they have started (each one being $\frac{2}{3}$ the preceding one), and there is never a last term. Let us consider the expression in parentheses in (1.2), namely

(1.3) $$\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\cdots.$$

This expression is an example of an *infinite series*, and we are asked to find its sum. Not all infinite series have sums; you can see that the series formed by adding the terms in (1.1a) does not have a finite sum. However, even when an infinite series does have a finite sum, we cannot find it by adding the terms because no matter how many we add there are always more. Thus we must find another method. (It is actually deeper than this; what we really have to do is to *define* what we mean by the sum of the series.)

Let us first find the sum of n terms in (1.3). The formula (Problem 2) for the sum of n terms of the geometric progression (1.1c) is

$$S_n = \frac{a(1-r^n)}{1-r}. \tag{1.4}$$

Using (1.4) in (1.3), we find

$$S_n = \frac{2}{3} + \frac{4}{9} + \cdots + \left(\frac{2}{3}\right)^n = \frac{\frac{2}{3}[1-(\frac{2}{3})^n]}{1-\frac{2}{3}} = 2\left[1-\left(\frac{2}{3}\right)^n\right]. \tag{1.5}$$

As n increases, $(\frac{2}{3})^n$ decreases and approaches zero. Then the sum of n terms approaches 2 as n increases, and we say that the sum of the series is 2. (This is really a definition: The sum of an infinite series is the limit of the sum of n terms as $n \to \infty$.) Then from (1.2), the total distance traveled by the ball is $1 + 2 \cdot 2 = 5$. This is an answer to a mathematical problem. A physicist might well object that a bounce the size of an atom is nonsense! However, after a number of bounces, the remaining infinite number of small terms contribute very little to the final answer (see Problem 1). Thus it makes little difference (in our answer for the total distance) whether we insist that the ball rolls after a certain number of bounces or whether we include the entire series, and it is easier to find the sum of the series than to find the sum of, say, twenty terms.

Series such as (1.3) whose terms form a geometric progression are called *geometric series.* We can write a geometric series in the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots . \tag{1.6}$$

The sum of the geometric series (if it has one) is by definition

$$S = \lim_{n\to\infty} S_n, \tag{1.7}$$

where S_n is the sum of n terms of the series. By following the method of the example above, you can show (Problem 2) that a geometric series has a sum if and only if $|r| < 1$, and in this case the sum is

$$S = \frac{a}{1-r}. \tag{1.8}$$

The series is then called *convergent.*

Here is an interesting use of (1.8). We can write $0.3333\cdots = \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \cdots = \frac{3/10}{1-1/10} = \frac{1}{3}$ by (1.8). Now of course you knew that, but how about $0.785714285714\cdots$? We can write this as $0.5+0.285714285714\cdots = \frac{1}{2} + \frac{0.285714}{1-10^{-6}} = \frac{1}{2} + \frac{285714}{999999} = \frac{1}{2} + \frac{2}{7} = \frac{11}{14}$. (Note that any repeating decimal is equivalent to a fraction which can be found by this method.) If you want to use a computer to do the arithmetic, be sure to tell it to give you an exact answer or it may hand you back the decimal you started with! You can also use a computer to sum the series, but using (1.8) may be simpler. (Also see Problem 14.)

PROBLEMS, SECTION 1

1. In the bouncing ball example above, find the height of the tenth rebound, and the distance traveled by the ball after it touches the ground the tenth time. Compare this distance with the total distance traveled.

2. Derive the formula (1.4) for the sum S_n of the geometric progression $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$. *Hint:* Multiply S_n by r and subtract the result from S_n; then solve for S_n. Show that the geometric series (1.6) converges if and only if $|r| < 1$; also show that if $|r| < 1$, the sum is given by equation (1.8).

Use equation (1.8) to find the fractions that are equivalent to the following repeating decimals:

3. $0.55555\cdots$ **4.** $0.818181\cdots$ **5.** $0.583333\cdots$

6. $0.61111\cdots$ **7.** $0.185185\cdots$ **8.** $0.694444\cdots$

9. $0.857142857142\cdots$ **10.** $0.576923076923076923\cdots$

11. $0.678571428571428571\cdots$

12. In a water purification process, one-nth of the impurity is removed in the first stage. In each succeeding stage, the amount of impurity removed is one-nth of that removed in the preceding stage. Show that if $n = 2$, the water can be made as pure as you like, but that if $n = 3$, at least one-half of the impurity will remain no matter how many stages are used.

13. If you invest a dollar at "6% interest compounded monthly," it amounts to $(1.005)^n$ dollars after n months. If you invest \$10 at the beginning of each month for 10 years (120 months), how much will you have at the end of the 10 years?

14. A computer program gives the result 1/6 for the sum of the series $\sum_{n=0}^{\infty}(-5)^n$. Show that this series is divergent. Do you see what happened? *Warning hint:* Always consider whether an answer is reasonable, whether it's a computer answer or your work by hand.

15. Connect the midpoints of the sides of an equilateral triangle to form 4 smaller equilateral triangles. Leave the middle small triangle blank, but for each of the other 3 small triangles, draw lines connecting the midpoints of the sides to create 4 tiny triangles. Again leave each middle tiny triangle blank and draw the lines to divide the others into 4 parts. Find the infinite series for the total area left blank if this process is continued indefinitely. (Suggestion: Let the area of the original triangle be 1; then the area of the first blank triangle is 1/4.) Sum the series to find the total area left blank. Is the answer what you expect? *Hint:* What is the "area" of a straight line? (Comment: You have constructed a *fractal* called the Sierpiński gasket. A fractal has the property that a magnified view of a small part of it looks very much like the original.)

16. Suppose a large number of particles are bouncing back and forth between $x = 0$ and $x = 1$, except that at each endpoint some escape. Let r be the fraction reflected each time; then $(1 - r)$ is the fraction escaping. Suppose the particles start at $x = 0$ heading toward $x = 1$; eventually all particles will escape. Write an infinite series for the fraction which escape at $x = 1$ and similarly for the fraction which escape at $x = 0$. Sum both the series. What is the largest fraction of the particles which can escape at $x = 0$? (Remember that r must be between 0 and 1.)

2. DEFINITIONS AND NOTATION

There are many other infinite series besides geometric series. Here are some examples:

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots, \tag{2.1a}$$

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \frac{4}{2^4} + \cdots, \tag{2.1b}$$

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots. \tag{2.1c}$$

In general, an infinite series means an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots, \tag{2.2}$$

where the a_n's (one for each positive integer n) are numbers or functions given by some formula or rule. The three dots in each case mean that the series never ends. The terms continue according to the law of formation, which is supposed to be evident to you by the time you reach the three dots. If there is apt to be doubt about how the terms are formed, a general or nth term is written like this:

$$1^2 + 2^2 + 3^2 + \cdots + n^2 + \cdots, \tag{2.3a}$$

$$x - x^2 + \frac{x^3}{2} + \cdots + \frac{(-1)^{n-1}x^n}{(n-1)!} + \cdots. \tag{2.3b}$$

(The quantity $n!$, read n factorial, means, for integral n, the product of all integers from 1 to n; for example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$. The quantity $0!$ is defined to be 1.) In (2.3a), it is easy to see without the general term that each term is just the square of the number of the term, that is, n^2. However, in (2.3b), if the formula for the general term were missing, you could probably make several reasonable guesses for the next term. To be sure of the law of formation, we must either know a good many more terms or have the formula for the general term. You should verify that the fourth term in (2.3b) is $-x^4/6$.

We can also write series in a shorter abbreviated form using a summation sign $\sum$ followed by the formula for the nth term. For example, (2.3a) would be written

$$1^2 + 2^2 + 3^2 + 4^2 + \cdots = \sum_{n=1}^{\infty} n^2 \tag{2.4}$$

(read "the sum of n^2 from $n = 1$ to ∞"). The series (2.3b) would be written

$$x - x^2 + \frac{x^3}{2} - \frac{x^4}{6} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}x^n}{(n-1)!}$$

For printing convenience, sums like (2.4) are often written $\sum_{n=1}^{\infty} n^2$.

In Section 1, we have mentioned both sequences and series. The lists in (1.1) are sequences; a *sequence* is simply a set of quantities, one for each n. A *series* is an indicated sum of such quantities, as in (1.3) or (1.6). We will be interested in various sequences related to a series: for example, the sequence a_n of terms of the series, the sequence S_n of partial sums [see (1.5) and (4.5)], the sequence R_n [see (4.7)], and the sequence ρ_n [see (6.2)]. In all these examples, we want to find the limit of a sequence as $n \to \infty$ (if the sequence has a limit). Although limits can be found by computer, many simple limits can be done faster by hand.

Example 1. Find the limit as $n \to \infty$ of the sequence

$$\frac{(2n-1)^4 + \sqrt{1+9n^8}}{1-n^3-7n^4}.$$

We divide numerator and denominator by n^4 and take the limit as $n \to \infty$. Then all terms go to zero except

$$\frac{2^4+\sqrt{9}}{-7} = -\frac{19}{7}.$$

Example 2. Find $\lim_{n\to\infty} \frac{\ln n}{n}$. By L'Hôpital's rule (see Section 15)

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1/n}{1} = 0.$$

Comment: Strictly speaking, we can't differentiate a function of n if n is an integer, but we can consider $f(x) = (\ln x)/x$, and the limit of the sequence is the same as the limit of $f(x)$.

Example 3. Find $\lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/n}$. We first find

$$\ln\left(\frac{1}{n}\right)^{1/n} = -\frac{1}{n}\ln n.$$

Then by Example 2, the limit of $(\ln n)/n$ is 0, so the original limit is $e^0 = 1$.

PROBLEMS, SECTION 2

In the following problems, find the limit of the given sequence as $n \to \infty$.

1. $\dfrac{n^2+5n^3}{2n^3+3\sqrt{4+n^6}}$
2. $\dfrac{(n+1)^2}{\sqrt{3+5n^2+4n^4}}$
3. $\dfrac{(-1)^n\sqrt{n+1}}{n}$
4. $\dfrac{2^n}{n^2}$
5. $\dfrac{10^n}{n!}$
6. $\dfrac{n^n}{n!}$
7. $(1+n^2)^{1/\ln n}$
8. $\dfrac{(n!)^2}{(2n)!}$
9. $n\sin(1/n)$

3. APPLICATIONS OF SERIES

In the example of the bouncing ball in Section 1, we saw that it is possible for the sum of an infinite series to be nearly the same as the sum of a fairly small number of terms at the beginning of the series (also see Problem 1.1). Many applied problems cannot be solved exactly, but we may be able to find an answer in terms of an infinite series, and then use only as many terms as necessary to obtain the needed accuracy. We shall see many examples of this both in this chapter and in later chapters. Differential equations (see Chapters 8 and 12) and partial differential equations (see Chapter 13) are frequently solved by using series. We will learn how to find series that represent functions; often a complicated function can be approximated by a few terms of its series (see Section 15).

But there is more to the subject of infinite series than making approximations. We will see (Chapter 2, Section 8) how we can use power series (that is, series whose terms are powers of x) to give meaning to functions of complex numbers, and (Chapter 3, Section 6) how to define a function of a matrix using the power series of the function. Also power series are just a first example of infinite series. In Chapter 7 we will learn about Fourier series (whose terms are sines and cosines). In Chapter 12, we will use power series to solve differential equations, and in Chapters 12 and 13, we will discuss other series such as Legendre and Bessel. Finally, in Chapter 14, we will discover how a study of power series clarifies our understanding of the mathematical functions we use in applications.

4. CONVERGENT AND DIVERGENT SERIES

We have been talking about series which have a finite sum. We have also seen that there are series which do not have finite sums, for example (2.1a). If a series has a finite sum, it is called *convergent*. Otherwise it is called *divergent*. It is important to know whether a series is convergent or divergent. Some weird things can happen if you try to apply ordinary algebra to a divergent series. Suppose we try it with the following series:

$$S = 1 + 2 + 4 + 8 + 16 + \cdots . \tag{4.1}$$

Then,

$$\begin{aligned} 2S &= 2 + 4 + 8 + 16 + \cdots = S - 1, \\ S &= -1. \end{aligned}$$

This is obvious nonsense, and you may laugh at the idea of trying to operate with such a violently divergent series as (4.1). But the same sort of thing can happen in more concealed fashion, and has happened and given wrong answers to people who were not careful enough about the way they used infinite series. At this point you probably would not recognize that the series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots \tag{4.2}$$

is divergent, but it is; and the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots \tag{4.3}$$

is convergent as it stands, but can be made to have *any* sum you like by combining the terms in a different order! (See Section 8.) You can see from these examples how essential it is to know whether a series converges, and also to know how to apply algebra to series correctly. There are even cases in which some divergent series can be used (see Chapter 11), but in this chapter we shall be concerned with convergent series.

Before we consider some tests for convergence, let us repeat the definition of convergence more carefully. Let us call the terms of the series a_n so that the series is

$$a_1 + a_2 + a_3 + a_4 + \cdots + a_n + \cdots. \tag{4.4}$$

Remember that the three dots mean that there is never a last term; the series goes on without end. Now consider the sums S_n that we obtain by adding more and more terms of the series. We define

$$\begin{aligned} S_1 &= a_1, \\ S_2 &= a_1 + a_2, \\ S_3 &= a_1 + a_2 + a_3, \\ &\cdots \\ S_n &= a_1 + a_2 + a_3 + \cdots + a_n. \end{aligned} \tag{4.5}$$

Each S_n is called a *partial sum*; it is the sum of the first n terms of the series. We had an example of this for a geometric progression in (1.4). The letter n can be any integer; for each n, S_n stops with the nth term. (Since S_n is not an infinite series, there is no question of convergence for it.) As n increases, the partial sums may increase without any limit as in the series (2.1a). They may oscillate as in the series $1-2+3-4+5-\cdots$ (which has partial sums $1, -1, 2, -2, 3, \cdots$) or they may have some more complicated behavior. One possibility is that the S_n's may, after a while, not change very much any more; the a_n's may become very small, and the S_n's come closer and closer to some value S. We are particularly interested in this case in which the S_n's approach a limiting value, say

$$\lim_{n\to\infty} S_n = S. \tag{4.6}$$

(It is understood that S is a finite number.) If this happens, we make the following definitions.

a. If the partial sums S_n of an infinite series tend to a limit S, the series is called *convergent*. Otherwise it is called *divergent*.

b. The limiting value S is called the *sum of the series*.

c. The difference $R_n = S - S_n$ is called the *remainder* (or the remainder after n terms). From (4.6), we see that

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} (S - S_n) = S - S = 0. \tag{4.7}$$

Example 1. We have already (Section 1) found S_n and S for a geometric series. From (1.8) and (1.4), we have for a geometric series, $R_n = \frac{ar^n}{1-r}$ which $\to 0$ as $n \to \infty$ if $|r| < 1$.

Example 2. By partial fractions, we can write $\frac{2}{n^2-1} = \frac{1}{n-1} - \frac{1}{n+1}$. Let's write out a number of terms of the series

$$\sum_{2}^{\infty} \frac{2}{n^2-1} = \sum_{2}^{\infty} \left(\frac{1}{n-1} - \frac{1}{n+1}\right) = \sum_{1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+2}\right)$$
$$= 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} + \frac{1}{4} - \frac{1}{6} + \frac{1}{5} - \frac{1}{7} + \frac{1}{6} - \frac{1}{8} + \cdots$$
$$+ \frac{1}{n-2} - \frac{1}{n} + \frac{1}{n-1} - \frac{1}{n+1} + \frac{1}{n} - \frac{1}{n+2} + \cdots.$$

Note the cancellation of terms; this kind of series is called a telescoping series. Satisfy yourself that when we have added the nth term $(\frac{1}{n} - \frac{1}{n+2})$, the only terms which have not cancelled are $1, \frac{1}{2}, \frac{-1}{n+1}$, and $\frac{-1}{n+2}$, so we have

$$S_n = \frac{3}{2} - \frac{1}{n+1} - \frac{1}{n+2}, \qquad S = \frac{3}{2}, \qquad R_n = \frac{1}{n+1} + \frac{1}{n+2}.$$

Example 3. Another interesting series is

$$\sum_{1}^{\infty} \ln\left(\frac{n}{n+1}\right) = \sum_{1}^{\infty} [\ln n - \ln(n+1)]$$
$$= \ln 1 - \ln 2 + \ln 2 - \ln 3 + \ln 3 - \ln 4 + \cdots + \ln n - \ln(n+1) \cdots.$$

Then $S_n = -\ln(n+1)$ which $\to -\infty$ as $n \to \infty$, so the series diverges. However, note that $a_n = \ln \frac{n}{n+1} \to \ln 1 = 0$ as $n \to \infty$, so we see that even if the terms tend to zero, a series may diverge.

PROBLEMS, SECTION 4

For the following series, write formulas for the sequences a_n, S_n, and R_n, and find the limits of the sequences as $n \to \infty$ (if the limits exist).

1. $\displaystyle\sum_{1}^{\infty} \frac{1}{2^n}$

2. $\displaystyle\sum_{0}^{\infty} \frac{1}{5^n}$

3. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} \cdots$

4. $\displaystyle\sum_{1}^{\infty} e^{-n \ln 3}$ *Hint:* What is $e^{-\ln 3}$?

5. $\displaystyle\sum_{0}^{\infty} e^{2n \ln \sin(\pi/3)}$ *Hint:* Simplify this.

6. $\displaystyle\sum_{1}^{\infty} \frac{1}{n(n+1)}$ *Hint:* $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.

7. $\frac{3}{1 \cdot 2} - \frac{5}{2 \cdot 3} + \frac{7}{3 \cdot 4} - \frac{9}{4 \cdot 5} + \cdots$

5. TESTING SERIES FOR CONVERGENCE; THE PRELIMINARY TEST

It is not in general possible to write a simple formula for S_n and find its limit as $n \to \infty$ (as we have done for a few special series), so we need some other way to find out whether a given series converges. Here we shall consider a few simple tests for convergence. These tests will illustrate some of the ideas involved in testing series for convergence and will work for a good many, but not all, cases. There are more complicated tests which you can find in other books. In some cases it may be quite a difficult mathematical problem to investigate the convergence of a complicated series. However, for our purposes the simple tests we give here will be sufficient.

First we discuss a useful *preliminary test.* In most cases you should apply this to a series before you use other tests.

> **Preliminary test.** If the terms of an infinite series do *not* tend to zero (that is, if $\lim_{n\to\infty} a_n \neq 0$), the series diverges. If $\lim_{n\to\infty} a_n = 0$, we must test further.

This is *not* a test for convergence; what it does is to weed out some very badly divergent series which you then do not have to spend time testing by more complicated methods. *Note carefully:* The preliminary test can *never* tell you that a series converges. It does *not* say that series converge if $a_n \to 0$ and, in fact, often they do not. A simple example is the harmonic series (4.2); the nth term certainly tends to zero, but we shall soon show that the series $\sum_{n=1}^{\infty} 1/n$ is divergent. On the other hand, in the series

$$\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \cdots$$

the terms are tending to 1, so by the preliminary test, this series diverges and no further testing is needed.

PROBLEMS, SECTION 5

Use the preliminary test to decide whether the following series are divergent or require further testing. *Careful:* Do *not* say that a series is convergent; the preliminary test cannot decide this.

1. $\frac{1}{2} - \frac{4}{5} + \frac{9}{10} - \frac{16}{17} + \frac{25}{26} - \frac{36}{37} + \cdots$

2. $\sqrt{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{4}}{3} + \frac{\sqrt{5}}{4} + \frac{\sqrt{6}}{5} + \cdots$

3. $\sum_{n=1}^{\infty} \frac{n+3}{n^2+10n}$

4. $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{(n+1)^2}$

5. $\sum_{n=1}^{\infty} \frac{n!}{n!+1}$

6. $\sum_{n=1}^{\infty} \frac{n!}{(n+1)!}$

7. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{\sqrt{n^3+1}}$

8. $\sum_{n=1}^{\infty} \frac{\ln n}{n}$

9. $\sum_{n=1}^{\infty} \frac{3^n}{2^n+3^n}$

10. $\sum_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right)$

11. Using (4.6), give a proof of the preliminary test. *Hint:* $S_n - S_{n-1} = a_n$.

6. CONVERGENCE TESTS FOR SERIES OF POSITIVE TERMS; ABSOLUTE CONVERGENCE

We are now going to consider four useful tests for series whose terms are all positive. If some of the terms of a series are negative, we may still want to consider the related series which we get by making all the terms positive; that is, we may consider the series whose terms are the absolute values of the terms of our original series. If this new series converges, we call the original series *absolutely convergent*. It can be proved that if a series converges absolutely, then it converges (Problem 7.9). This means that if the series of absolute values converges, the series is still convergent when you put back the original minus signs. (The sum is different, of course.) The following four tests may be used, then, either for testing series of positive terms, or for testing any series for absolute convergence.

A. The Comparison Test

This test has two parts, (a) and (b).

(a) Let

$$m_1 + m_2 + m_3 + m_4 + \cdots$$

be a series of positive terms which you know converges. Then the series you are testing, namely

$$a_1 + a_2 + a_3 + a_4 + \cdots$$

is absolutely convergent if $|a_n| \le m_n$ (that is, if the absolute value of each term of the a series is no larger than the corresponding term of the m series) for all n from some point on, say after the third term (or the millionth term). See the example and discussion below.

(b) Let

$$d_1 + d_2 + d_3 + d_4 + \cdots$$

be a series of positive terms which you know diverges. Then the series

$$|a_1| + |a_2| + |a_3| + |a_4| + \cdots$$

diverges if $|a_n| \ge d_n$ for all n from some point on.

Warning: Note carefully that neither $|a_n| \ge m_n$ nor $|a_n| \le d_n$ tells us anything. That is, if a series has terms larger than those of a convergent series, it may still converge or it may diverge—we must test it further. Similarly, if a series has terms smaller than those of a divergent series, it may still diverge, or it may converge.

Example. Test $\sum_{n=1}^{\infty} \frac{1}{n!} = 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots$ for convergence.

As a comparison series, we choose the geometric series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots.$$

Notice that we do not care about the first few terms (or, in fact, any finite number of terms) in a series, because they can affect the sum of the series but *not* whether it converges. When we ask whether a series converges or not, we are asking what happens as we add more and more terms for larger and larger n. Does the sum increase indefinitely, or does it approach a limit? What the first five or hundred or million terms are has no effect on whether the sum eventually increases indefinitely or approaches a limit. Consequently we frequently ignore some of the early terms in testing series for convergence.

In our example, the terms of $\sum_{n=1}^{\infty} 1/n!$ are smaller than the corresponding terms of $\sum_{n=1}^{\infty} 1/2^n$ for all $n > 3$ (Problem 1). We know that the geometric series converges because its ratio is $\frac{1}{2}$. Therefore $\sum_{n=1}^{\infty} 1/n!$ converges also.

$\Rightarrow \frac{1}{n!} \to 0,\ n \to \infty$. Of course.

PROBLEMS, SECTION 6

1. Show that $n! > 2^n$ for all $n > 3$. *Hint:* Write out a few terms; then consider what you multiply by to go from, say, 5! to 6! and from 2^5 to 2^6.

2. Prove that the harmonic series $\sum_{n=1}^{\infty} 1/n$ is divergent by comparing it with the series

$$1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\left(8 \text{ terms each equal to } \frac{1}{16}\right)+\cdots,$$

$$\text{which is}\quad 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\cdots.$$

3. Prove the convergence of $\sum_{n=1}^{\infty} 1/n^2$ by grouping terms somewhat as in Problem 2. Not done!

4. Use the comparison test to prove the convergence of the following series:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n+3^n}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\,2^n}$

$n2^n > 2^n$, $\frac{1}{n2^n} < \frac{1}{2^n}$: geometric $r = 1/2$

5. Test the following series for convergence using the comparison test.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ *Hint:* Which is larger, n or $\sqrt{n}$? (b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$

6. There are 9 one-digit numbers (1 to 9), 90 two-digit numbers (10 to 99). How many three-digit, four-digit, etc., numbers are there? The first 9 terms of the harmonic series $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{9}$ are all greater than $\frac{1}{10}$; similarly consider the next 90 terms, and so on. Thus prove the divergence of the harmonic series by comparison with the series

$$\left[\tfrac{1}{10}+\tfrac{1}{10}+\cdots(9 \text{ terms each} = \tfrac{1}{10})\right]+\left[\,90 \text{ terms each} = \tfrac{1}{100}\right]+\cdots$$
$$=\tfrac{9}{10}+\tfrac{90}{100}+\cdots=\tfrac{9}{10}+\tfrac{9}{10}+\cdots.$$

The comparison test is really the basic test from which other tests are derived. It is probably the most useful test of all for the experienced mathematician but it is often hard to think of a satisfactory m series until you have had a good deal of experience with series. Consequently, you will probably not use it as often as the next three tests.

B. The Integral Test

We can use this test when the terms of the series are positive and not increasing, that is, when $a_{n+1} \le a_n$. (Again remember that we can ignore any finite number of terms of the series; thus the test can still be used even if the condition $a_{n+1} \le a_n$ does not hold for a finite number of terms.) To apply the test we think of a_n as a function of the variable n, and, forgetting our previous meaning of n, we allow it to take all values, not just integral ones. The test states that:

If $0 < a_{n+1} \le a_n$ for $n > N$, then $\sum^{\infty} a_n$ converges if $\int^{\infty} a_n \, dn$ is finite and diverges if the integral is infinite. (The integral is to be evaluated *only* at the upper limit; no lower limit is needed.)

To understand this test, imagine a graph sketched of a_n as a function of n. For example, in testing the harmonic series $\sum_{n=1}^{\infty} 1/n$, we consider the graph of the function $y = 1/n$ (similar to Figures 6.1 and 6.2) letting n have all values, not just integral ones. Then the values of y on the graph at $n = 1, 2, 3, \cdots$, are the terms of the series. In Figures 6.1 and 6.2, the areas of the rectangles are just the terms of the series. Notice that in Figure 6.1 the top edge of each rectangle is above the curve, so that the area of the rectangles is greater than the corresponding area under the curve. On the other hand, in Figure 6.2 the rectangles lie below the curve, so their area is less than the corresponding area under the curve. Now the areas of the rectangles are just the terms of the series, and the area under the curve is an integral of $y \, dn$ or $a_n \, dn$. The upper limit on the integrals is ∞ and the lower limit could be made to correspond to any term of the series we wanted to start with. For example (see Figure 6.1), $\int_3^{\infty} a_n \, dn$ is less than the sum of the series from a_3 on, but (see Figure 6.2) greater than the sum of the series from a_4 on. If the integral is finite, then the sum of the series from a_4 on is finite, that is, the series converges. Note again that the terms at the beginning of a series have nothing to do with convergence. On the other hand, if the integral is infinite, then the sum of the series from a_3 on is infinite and the series diverges. Since the beginning terms are of no interest, you should simply evaluate $\int^{\infty} a_n \, dn$. (Also see Problem 16.)

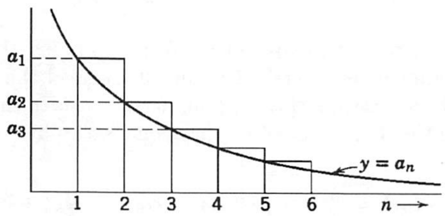

Figure 6.1

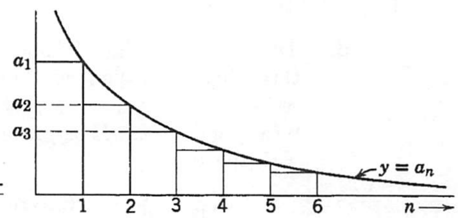

Figure 6.2

▶ **Example.** Test for convergence the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots . \tag{6.1}$$

Using the integral test, we evaluate

$$\int^{\infty} \frac{1}{n} \, dn = \ln n \Big|^{\infty} = \infty.$$

(We use the symbol ln to mean a natural logarithm, that is, a logarithm to the base e.) Since the integral is infinite, the series diverges.

PROBLEMS, SECTION 6

Use the integral test to find whether the following series converge or diverge. *Hint* and *warning:* Do *not* use lower limits on your integrals (see Problem 16).

7. $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

8. $\sum_{n=1}^{\infty} \frac{n}{n^2+4}$

9. $\sum_{n=3}^{\infty} \frac{1}{n^2-4}$

10. $\sum_{n=1}^{\infty} \frac{e^n}{e^{2n}+9}$

11. $\sum_{1}^{\infty} \frac{1}{n(1+\ln n)^{3/2}}$

12. $\sum_{1}^{\infty} \frac{n}{(n^2+1)^2}$

13. $\sum_{1}^{\infty} \frac{n^2}{n^3+1}$

14. $\sum_{1}^{\infty} \frac{1}{\sqrt{n^2+9}}$

15. Use the integral test to prove the following so-called p-series test. The series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \quad \text{is} \quad \begin{cases} \text{convergent} & \text{if } p > 1, \\ \text{divergent} & \text{if } p \le 1. \end{cases}$$

Caution: Do $p = 1$ separately.

16. In testing $\sum 1/n^2$ for convergence, a student evaluates $\int_0^\infty n^{-2} dn = -n^{-1}|_0^\infty = 0 + \infty = \infty$ and concludes (erroneously) that the series diverges. What is wrong? *Hint:* Consider the area under the curve in a diagram such as Figure 6.1 or 6.2. This example shows the danger of using a lower limit in the integral test.

17. Use the integral test to show that $\sum_{n=0}^{\infty} e^{-n^2}$ converges. *Hint:* Although you cannot *evaluate* the integral, you can show that it is finite (which is all that is necessary) by comparing it with $\int^\infty e^{-n} dn$.

C. The Ratio Test

The integral test depends on your being able to integrate $a_n dn$; this is not always easy! We consider another test which will handle many cases in which we cannot evaluate the integral. Recall that in the geometric series each term could be obtained by multiplying the one before it by the ratio r, that is, $a_{n+1} = ra_n$ or $a_{n+1}/a_n = r$. For other series the ratio a_{n+1}/a_n is not constant but depends on n; let us call the absolute value of this ratio ρ_n. Let us also find the limit (if there is one) of the sequence ρ_n as $n \to \infty$ and call this limit ρ. Thus we define ρ_n and ρ by the equations

$$\rho_n = \left| \frac{a_{n+1}}{a_n} \right|, \qquad \rho = \lim_{n\to\infty} \rho_n. \tag{6.2}$$

If you recall that a geometric series converges if $|r| < 1$, it may seem plausible that a series with $\rho < 1$ should converge and this is true. This statement can be proved (Problem 30) by comparing the series to be tested with a geometric series. Like a geometric series with $|r| > 1$, a series with $\rho > 1$ also diverges (Problem 30). However, if $\rho = 1$, the ratio test does not tell us anything; some series with $\rho = 1$ converge and some diverge, so we must find another test (say one of the two preceding tests). To summarize the ratio test:

$$\text{(6.3)} \qquad \text{If } \begin{cases} \rho < 1, & \text{the series converges;} \\ \rho = 1, & \text{use a different test;} \\ \rho > 1, & \text{the series diverges.} \end{cases}$$

Example 1. Test for convergence the series

$$1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots.$$

Using (6.2), we have

$$\rho_n = \left| \frac{1}{(n+1)!} \div \frac{1}{n!} \right|$$
$$= \frac{n!}{(n+1)!} = \frac{n(n-1)\cdots 3\cdot 2\cdot 1}{(n+1)(n)(n-1)\cdots 3\cdot 2\cdot 1} = \frac{1}{n+1},$$
$$\rho = \lim_{n\to\infty} \rho_n = \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

Since $\rho < 1$, the series converges.

Example 2. Test for convergence the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots.$$

We find

$$\rho_n = \left| \frac{1}{n+1} \div \frac{1}{n} \right| = \frac{n}{n+1},$$
$$\rho = \lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1.$$

Here the test tells us nothing and we must use some different test. A word of warning from this example: Notice that $\rho_n = n/(n+1)$ is always less than 1. Be careful not to confuse this ratio with ρ and conclude incorrectly that this series converges. (It is actually divergent as we proved by the integral test.) Remember that ρ is *not* the same as the ratio $\rho_n = |a_{n+1}/a_n|$, but is the *limit* of this ratio as $n \to \infty$.

PROBLEMS, SECTION 6

Use the ratio test to find whether the following series converge or diverge:

18. $\sum_{n=1}^{\infty} \frac{2^n}{n^2}$ **19.** $\sum_{n=0}^{\infty} \frac{3^n}{2^{2n}}$ **20.** $\sum_{n=0}^{\infty} \frac{n!}{(2n)!}$

21. $\sum_{n=0}^{\infty} \frac{5^n (n!)^2}{(2n)!}$ **22.** $\sum_{n=1}^{\infty} \frac{10^n}{(n!)^2}$ **23.** $\sum_{n=1}^{\infty} \frac{n!}{100^n}$

24. $\sum_{n=0}^{\infty} \frac{3^{2n}}{2^{3n}}$ **25.** $\sum_{n=0}^{\infty} \frac{e^n}{\sqrt{n!}}$ **26.** $\sum_{n=0}^{\infty} \frac{(n!)^3 e^{3n}}{(3n)!}$

27. $\sum_{n=0}^{\infty} \frac{100^n}{n^{200}}$ **28.** $\sum_{n=0}^{\infty} \frac{n!(2n)!}{(3n)!}$ **29.** $\sum_{n=0}^{\infty} \frac{\sqrt{(2n)!}}{n!}$

30. Prove the ratio test. *Hint:* If $|a_{n+1}/a_n| \to \rho < 1$, take σ so that $\rho < \sigma < 1$. Then $|a_{n+1}/a_n| < \sigma$ if n is large, say $n \geq N$. This means that we have $|a_{N+1}| < \sigma|a_N|$, $|a_{N+2}| < \sigma|a_{N+1}| < \sigma^2|a_N|$, and so on. Compare with the geometric series

$$\sum_{n=1}^{\infty} \sigma^n |a_N|.$$

Also prove that a series with $\rho > 1$ diverges. *Hint:* Take $\rho > \sigma > 1$, and use the preliminary test.

D. A Special Comparison Test

This test has two parts: (a) a convergence test, and (b) a divergence test. (See Problem 37.)

(a) If $\sum_{n=1}^{\infty} b_n$ is a convergent series of positive terms and $a_n \geq 0$ and a_n/b_n tends to a (finite) limit, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\sum_{n=1}^{\infty} d_n$ is a divergent series of positive terms and $a_n \geq 0$ and a_n/d_n tends to a limit greater than 0 (or tends to $+\infty$), then $\sum_{n=1}^{\infty} a_n$ diverges.

There are really two steps in using either of these tests, namely, to decide on a comparison series, and then to compute the required limit. The first part is the most important; given a good comparison series it is a routine process to find the needed limit. The method of finding the comparison series is best shown by examples.

Example 1. Test for convergence

$$\sum_{n=3}^{\infty} \frac{\sqrt{2n^2 - 5n + 1}}{4n^3 - 7n^2 + 2}.$$

Remember that whether a series converges or diverges depends on what the terms are as n becomes larger and larger. We are interested in the nth term as $n \to \infty$. Think of $n = 10^{10}$ or 10^{100}, say; a little calculation should convince you that as n increases, $2n^2 - 5n + 1$ is $2n^2$ to quite high accuracy. Similarly, the denominator in our example is nearly $4n^3$ for large n. By Section 9, fact 1, we see that the factor $\sqrt{2}/4$ in every term does not affect convergence. So we consider as a comparison series just

$$\sum_{n=3}^{\infty} \frac{\sqrt{n^2}}{n^3} = \sum_{n=3}^{\infty} \frac{1}{n^2}$$

which we recognize (say by integral test) as a convergent series. Hence we use test (a) to try to show that the given series converges. We have:

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\left(\frac{\sqrt{2n^2-5n+1}}{4n^3-7n^2+2}\div\frac{1}{n^2}\right)$$
$$=\lim_{n\to\infty}\frac{n^2\sqrt{2n^2-5n+1}}{4n^3-7n^2+2}$$
$$=\lim_{n\to\infty}\frac{\sqrt{2-\frac{5}{n}+\frac{1}{n^2}}}{4-\frac{7}{n}+\frac{2}{n^3}}=\frac{\sqrt{2}}{4}.$$

Since this is a finite limit, the given series converges. (With practice, you won't need to do all this algebra! You should be able to look at the original problem and see that, for large n, the terms are essentially $1/n^2$, so the series converges.)

▶ **Example 2.** Test for convergence

$$\sum_{n=2}^{\infty}\frac{3^n-n^3}{n^5-5n^2}.$$

Here we must first decide which is the important term as $n\to\infty$; is it 3^n or n^3? We can find out by comparing their logarithms since $\ln N$ and N increase or decrease together. We have $\ln 3^n = n\ln 3$, and $\ln n^3 = 3\ln n$. Now $\ln n$ is much smaller than n, so for large n we have $n\ln 3 > 3\ln n$, and $3^n > n^3$. (You might like to compute $100^3 = 10^6$, and $3^{100} > 5\times 10^{47}$.) The denominator of the given series is approximately n^5. Thus the comparison series is $\sum_{n=2}^{\infty} 3^n/n^5$. It is easy to prove this divergent by the ratio test. Now by test (b)

$$\lim_{n\to\infty}\left(\frac{3^n-n^3}{n^5-5n^2}\div\frac{3^n}{n^5}\right)=\lim_{n\to\infty}\frac{1-\frac{n^3}{3^n}}{1-\frac{5}{n^3}}=1$$

which is greater than zero, so the given series diverges.

▶ PROBLEMS, SECTION 6

Use the special comparison test to find whether the following series converge or diverge.

31. $\displaystyle\sum_{n=9}^{\infty}\frac{(2n+1)(3n-5)}{\sqrt{n^2-73}}$

32. $\displaystyle\sum_{n=0}^{\infty}\frac{n(n+1)}{(n+2)^2(n+3)}$

33. $\displaystyle\sum_{n=5}^{\infty}\frac{1}{2^n-n^2}$

34. $\displaystyle\sum_{n=1}^{\infty}\frac{n^2+3n+4}{n^4+7n^3+6n-3}$

35. $\displaystyle\sum_{n=3}^{\infty}\frac{(n-\ln n)^2}{5n^4-3n^2+1}$

36. $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt{n^3+5n-1}}{n^2-\sin n^3}$

37. Prove the special comparison test. *Hint* (part a): If $a_n/b_n\to L$ and $M>L$, then $a_n < Mb_n$ for large n. Compare $\sum_{n=1}^{\infty} a_n$ with $\sum_{n=1}^{\infty} Mb_n$.

7. ALTERNATING SERIES

So far we have been talking about series of positive terms (including series of absolute values). Now we want to consider one important case of a series whose terms have mixed signs. An *alternating series* is a series whose terms are alternately plus and minus; for example,

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots+\frac{(-1)^{n+1}}{n}+\cdots \tag{7.1}$$

is an alternating series. We ask two questions about an alternating series. Does it converge? Does it converge absolutely (that is, when we make all signs positive)? Let us consider the second question first. In this example the series of absolute values

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}+\cdots$$

is the harmonic series (6.1), which diverges. We say that the series (7.1) is not absolutely convergent. Next we must ask whether (7.1) converges as it stands. If it had turned out to be absolutely convergent, we would not have to ask this question since an absolutely convergent series is also convergent (Problem 9). However, a series which is not absolutely convergent may converge or it may diverge; we must test it further. For alternating series the test is very simple:

Test for alternating series. An alternating series converges if the absolute value of the terms decreases steadily to zero, that is, if $|a_{n+1}| \le |a_n|$ and $\lim_{n\to\infty} a_n = 0$.

In our example $\frac{1}{n+1} < \frac{1}{n}$, and $\lim_{n\to\infty} \frac{1}{n} = 0$, so (7.1) converges.

PROBLEMS, SECTION 7

Test the following series for convergence.

1. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

2. $\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$

3. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

4. $\sum_{n=1}^{\infty} \frac{(-3)^n}{n!}$

5. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

6. $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+5}$

7. $\sum_{n=0}^{\infty} \frac{(-1)^n n}{1+n^2}$

8. $\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{10n}}{n+2}$

9. Prove that an absolutely convergent series $\sum_{n=1}^{\infty} a_n$ is convergent. *Hint:* Put $b_n = a_n + |a_n|$. Then the b_n are nonnegative; we have $|b_n| \le 2|a_n|$ and $a_n = b_n - |a_n|$.

10. The following alternating series are divergent (but you are not asked to prove this). Show that $a_n \to 0$. Why doesn't the alternating series test prove (incorrectly) that these series converge?

(a) $2-\frac{1}{2}+\frac{2}{3}-\frac{1}{4}+\frac{2}{5}-\frac{1}{6}+\frac{2}{7}-\frac{1}{8}\cdots$

(b) $\frac{1}{\sqrt{2}}-\frac{1}{2}+\frac{1}{\sqrt{3}}-\frac{1}{3}+\frac{1}{\sqrt{4}}-\frac{1}{4}+\frac{1}{\sqrt{5}}-\frac{1}{5}\cdots$

8. CONDITIONALLY CONVERGENT SERIES

A series like (7.1) which converges, but does not converge absolutely, is called *conditionally convergent.* You have to use special care in handling conditionally convergent series because the positive terms alone form a divergent series and so do the negative terms alone. If you rearrange the terms, you will probably change the sum of the series, and you may even make it diverge! It is possible to rearrange the terms to make the sum any number you wish. Let us do this with the alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$. Suppose we want to make the sum equal to 1.5. First we take enough positive terms to add to just over 1.5. The first three positive terms do this:

$$1 + \frac{1}{3} + \frac{1}{5} = 1\frac{8}{15} > 1.5.$$

Then we take enough negative terms to bring the partial sum back under 1.5; the one term $-\frac{1}{2}$ does this. Again we add positive terms until we have a little more than 1.5, and so on. Since the terms of the series are decreasing in absolute value, we are able (as we continue this process) to get partial sums just a little more or a little less than 1.5 but always nearer and nearer to 1.5. But this is what convergence of the series to the sum 1.5 means: that the partial sums should approach 1.5. You should see that we could pick in advance *any* sum that we want, and rearrange the terms of this series to get it. Thus, we must not rearrange the terms of a conditionally convergent series since its convergence and its sum depend on the fact that the terms are added in a particular order.

Here is a physical example of such a series which emphasizes the care needed in applying mathematical approximations in physical problems. Coulomb's law in electricity says that the force between two charges is equal to the product of the charges divided by the square of the distance between them (in electrostatic units; to use other units, say SI, we need only multiply by a numerical constant). Suppose there are unit positive charges at $x = 0, \sqrt{2}, \sqrt{4}, \sqrt{6}, \sqrt{8}, \cdots$, and unit negative charges at $x = 1, \sqrt{3}, \sqrt{5}, \sqrt{7}, \cdots$. We want to know the total force acting on the unit positive charge at $x = 0$ due to all the other charges. The negative charges attract the charge at $x = 0$ and try to pull it to the right; we call the forces exerted by them positive, since they are in the direction of the positive x axis. The forces due to the positive charges are in the negative x direction, and we call them negative. For example, the force due to the positive charge at $x = \sqrt{2}$ is $-(1 \cdot 1)/(\sqrt{2})^2 = -1/2$. The total force on the charge at $x = 0$ is, then,

$$F = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots. \tag{8.1}$$

Now we know that this series converges as it stands (Section 7). But we have also seen that its sum (even the fact that it converges) can be changed by rearranging the terms. Physically this means that the force on the charge at the origin depends not only on the size and position of the charges, but also on the *order* in which we place them in their positions! This may very well go strongly against your physical intuition. You feel that a physical problem like this should have a definite answer. Think of it this way. Suppose there are two crews of workers, one crew placing the positive charges and one placing the negative. If one crew works faster than the other, it is clear that the force at any stage may be far from the F of equation (8.1) because there are many extra charges of one sign. The crews can never place *all* the charges because there are an infinite number of them. At any stage the forces which would arise from the positive charges that are not yet in place, form a divergent series; similarly, the forces due to the unplaced negative charges form a divergent series of the opposite sign. We cannot then stop at some point and say that the rest of the series is negligible as we could in the bouncing ball problem in Section 1. But if we specify the *order* in which the charges are to be placed, then the sum S of the series is determined (S is probably different from F in (8.1) unless the charges are placed alternately). Physically this means that the value of the force as the crews proceed comes closer and closer to S, and we can use the sum of the (properly arranged) *infinite* series as a good approximation to the force.

9. USEFUL FACTS ABOUT SERIES

We state the following facts for reference:

1. The convergence or divergence of a series is not affected by multiplying every term of the series by the same nonzero constant. Neither is it affected by changing a finite number of terms (for example, omitting the first few terms).

2. Two convergent series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ may be added (or subtracted) term by term. (Adding "term by term" means that the nth term of the sum is $a_n + b_n$.) The resulting series is convergent, and its sum is obtained by adding (subtracting) the sums of the two given series.

3. The terms of an *absolutely convergent series* may be rearranged in any order without affecting either the convergence or the sum. This is *not true* of conditionally convergent series as we have seen in Section 8.

PROBLEMS, SECTION 9 Homework

Test the following series for convergence or divergence. Decide for yourself which test is easiest to use, but don't forget the preliminary test. Use the facts stated above when they apply.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{(n+2)(n+3)}$ **2.** $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+1}$ **3.** $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{\ln 3}}$

4. $\displaystyle\sum_{n=0}^{\infty} \frac{n^2}{n^3+4}$ **5.** $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3-4}$ **6.** $\displaystyle\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!}$

7. $\displaystyle\sum_{n=0}^{\infty} \frac{(2n)!}{3^n(n!)^2}$ **8.** $\displaystyle\sum_{n=1}^{\infty} \frac{n^5}{5^n}$ **9.** $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}$

10. $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{n}{n-1}$ **11.** $\displaystyle\sum_{n=4}^{\infty} \frac{2n}{n^2-9}$ **12.** $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2-n}$

13. $\displaystyle\sum_{n=0}^{\infty} \frac{n}{(n^2+4)^{3/2}}$ **14.** $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2-n}$ **15.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n!}{10^n}$

16. $\displaystyle\sum_{n=0}^{\infty} \frac{2+(-1)^n}{n^2+7}$ **17.** $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^3}{(3n)!}$ **18.** $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{2^{\ln n}}$

19. $\frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{2^3} - \frac{1}{3^3} + \frac{1}{2^4} - \frac{1}{3^4} + \cdots$

20. $\frac{1}{2} + \frac{1}{2^2} - \frac{1}{3} - \frac{1}{3^2} + \frac{1}{4} + \frac{1}{4^2} - \frac{1}{5} - \frac{1}{5^2} + \cdots$

21. $\sum_{n=1}^{\infty} a_n$ if $a_{n+1} = \frac{n}{2n+3} a_n$

Tema 1

22. (a) $\sum_{n=1}^{\infty} \frac{1}{3^{\ln n}}$ (b) $\sum_{n=1}^{\infty} \frac{1}{2^{\ln n}}$

(c) For what values of k is $\sum_{n=1}^{\infty} \frac{1}{k^{\ln n}}$ convergent?

Tema 2

10. POWER SERIES; INTERVAL OF CONVERGENCE

We have been discussing series whose terms were constants. Even more important and useful are series whose terms are functions of x. There are many such series, but in this chapter we shall consider series in which the nth term is a constant times x^n or a constant times $(x-a)^n$ where a is a constant. These are called *power series*, because the terms are multiples of powers of x or of $(x-a)$. In later chapters we shall consider Fourier series whose terms involve sines and cosines, and other series (Legendre, Bessel, etc.) in which the terms may be polynomials or other functions.

By definition, a power series is of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \quad \text{or} \tag{10.1}$$
$$\sum_{n=0}^{\infty} a_n (x-a)^n = a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 + \cdots,$$

where the coefficients a_n are constants. Here are some examples:

$$1 - \frac{x}{2} + \frac{x^2}{4} - \frac{x^3}{8} + \cdots + \frac{(-x)^n}{2^n} + \cdots, \tag{10.2a}$$

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n+1}x^n}{n} + \cdots, \tag{10.2b}$$

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^{n+1}x^{2n-1}}{(2n-1)!} + \cdots, \tag{10.2c}$$

$$1 + \frac{(x+2)}{\sqrt{2}} + \frac{(x+2)^2}{\sqrt{3}} + \cdots + \frac{(x+2)^n}{\sqrt{n+1}} + \cdots. \tag{10.2d}$$

Whether a power series converges or not depends on the value of x we are considering. We often use the ratio test to find the values of x for which a series converges. We illustrate this by testing each of the four series (10.2). Recall that in the ratio test we divide term $n+1$ by term n and take the absolute value of this ratio to get ρ_n, and then take the limit of ρ_n as $n \to \infty$ to get ρ.

Example 1. For (10.2a), we have

$$\rho_n = \left| \frac{(-x)^{n+1}}{2^{n+1}} \div \frac{(-x)^n}{2^n} \right| = \left| \frac{x}{2} \right|,$$
$$\rho = \left| \frac{x}{2} \right|.$$